

BASIC AND DEGENERATE PREGEOMETRIES

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ABSTRACT. We study pairs (Γ, G) , where Γ is a ‘Buekenhout–Tits’ pregeometry with all rank 2 truncations connected, and $G \leq \text{Aut } \Gamma$ is transitive on the set of elements of each type. The family of such pairs is closed under forming quotients with respect to G -invariant type-refining partitions of the element set of Γ . We identify the ‘basic’ pairs (those that admit no non-degenerate quotients), and show, by studying quotients and direct decompositions, that the study of basic pregeometries reduces to examining those where the group G is faithful and primitive on the set of elements of each type. We also study the special case of normal quotients, where we take quotients with respect to the orbits of a normal subgroup of G . There is a similar reduction for normal-basic pregeometries to those where G is faithful and quasiprimitive on the set of elements of each type.

1. INTRODUCTION

In the 1950s, Jacques Tits [15, 16, 17] gave a unified geometric interpretation for all complex Lie groups, and in particular for the exceptional Lie groups, by means of a new kind of geometry, later called a building. According to Tits, ‘it was really the geometric interpretations of these mysterious groups, the exceptional groups, that triggered everything’ [14, page 474]. In the 1970s, inspired by Tits’ work, Francis Buekenhout studied a more general family of incidence geometries aiming to find a geometrical interpretation for all the sporadic finite simple groups which, in conjunction with Tits’ geometries, would provide a unified theory for all the (then known) finite simple groups, as well as a ‘unification of the known classes of combinatorial geometries’. As Buekenhout explained further (see [3]), his approach ‘reduces combinatorial geometry to graph theory and lattice theory’.

In this paper we analyse a larger family of Buekenhout–Tits geometries that satisfy mild combinatorial properties (much weaker than buildings) and correspondingly mild transitivity properties (much weaker than flag-transitivity). We seek a rationale that enables us to identify the fundamental or ‘basic’ geometries in the family, and to determine what additional geometric and symmetry properties these basic geometries possess. In what might be regarded as a pilot for this research project, we (the first,

second and fourth authors) studied the family of connected rank 2, locally 2-transitive geometries (or in graph theoretic language, locally s -arc transitive graphs), identifying and describing the basic examples and their automorphism groups, see [7, 8, 9, 10, 11]. We wanted to know if a similar combinatorial/geometrical approach would identify the basic geometries in the larger family as some (tractible) class containing the geometries for simple groups as a natural subclass.

The incidence geometries we consider are now called pregeometries, though Tits called them geometries in [17]. Pregeometries, their rank 2 truncations, and their automorphism groups, are defined in Subsection 2.1. We study the family \mathcal{G} of pairs (Γ, G) , where Γ is a pregeometry with all rank 2 truncations connected, and $G \leq \text{Aut } \Gamma$ is transitive on the set of elements of each type. The family \mathcal{G} is closed under taking certain kinds of quotients, and our aim is to understand \mathcal{G} by first determining the ‘basic’ pairs in \mathcal{G} (those that admit no non-degenerate quotients) and then describing how all other pairs in \mathcal{G} are derived from these ‘basic’ ones.

We consider two kinds of quotient operations on pairs $(\Gamma, G) \in \mathcal{G}$, namely *imprimitive* quotients, where we take quotients with respect to G -invariant type-refining partitions of the element set of Γ , and *normal* quotients where we take quotients with respect to the orbits of a normal subgroup of G (the latter is a special case of the former). We find that it is appropriate to use slightly different definitions of degeneracy according to whether we are considering imprimitive or normal quotients, and this leads to correspondingly different concepts of ‘basic’. Thus we have respectively the concepts *G -primitive-basic* and *G -normal-basic* (defined formally in Subsections 3.1 and 4.1 respectively).

Theorem 1.1 shows that the study of G -primitive-basic pregeometries reduces to examining those where, for each type in the pregeometry, G is faithful and primitive on the set of elements of that type. Theorem 1.2 shows that there is a similar reduction for G -normal-basic pregeometries to those where G is faithful and quasiprimitive on the set of elements of each type. (A permutation group is called *quasiprimitive* if every non-trivial normal subgroup is transitive; it is *primitive* if the only invariant partitions are trivial.) Since a pregeometry can be viewed as a multipartite graph, and the automorphism group of a pregeometry is the stabiliser in the graph automorphism group of each part in the multipartition, these results also provide information about automorphism groups of multipartite graphs.

Some notation: Given two pregeometries Γ_1 and Γ_2 , the *direct sum* $\Gamma_1 \oplus \Gamma_2$ is (loosely speaking) the disjoint union of the two pregeometries with complete incidence between the two element sets (see Section 2.4). A pregeometry is called *indecomposable* if it is not the direct sum of smaller pregeometries. We denote by \mathcal{Q} the subset of all pairs $(\Gamma, G) \in \mathcal{G}$ such that G is faithful and quasiprimitive on the set of elements of each type; and we use $\Gamma(k, m)$ to denote the pregeometry whose incidence graph is the complete multipartite graph $K_{k \times m}$ with k parts of size m . These (and other)

special terms and notation in the theorems are explained further in Sections 2.1 and 2.4 and just before Proposition 4.3.

Theorem 1.1. *Let $(\Gamma, G) \in \mathcal{G}$ such that Γ is G -primitive-basic. Then Γ admits a unique decomposition $\Gamma_1 \oplus \dots \oplus \Gamma_\ell$ where for each i , Γ_i is indecomposable and the group G^{Γ_i} induced on Γ_i is faithful and primitive on the set of elements of each type in Γ_i .*

Theorem 1.2. *Let $(\Gamma, G) \in \mathcal{G}$ such that Γ is G -normal-basic of rank k . Then exactly one of the following holds for some integers m, m' .*

- (i) $(\Gamma, G) \in \mathcal{Q}$;
- (ii) $\Gamma = \Gamma_0 \oplus \Gamma'$ where $(\Gamma_0, G^{\Gamma_0}) \in \mathcal{Q}$, $G \cong G^{\Gamma_0}$, and either
 - (a) Γ_0 has rank $k - 1 \geq 1$, and $\Gamma' = \Gamma(1, m)$, or
 - (b) Γ_0 has rank $k - 2 \geq 2$, and $\Gamma' = \Gamma(1, m) \oplus \Gamma(1, m')$;
- (iii) $k = 2$ and $\Gamma = \Gamma(1, m) \oplus \Gamma(1, m')$; or $k = 3$ and $\Gamma = \Gamma(1, m) \oplus \Gamma(1, m/a) \oplus \Gamma(1, m/b)$ for some a, b dividing m ; or $\Gamma = \Gamma(k, m)$ with $3 \leq k \leq m + 1$;
- (iv) Γ has a rank $k - 1$ truncation Γ_0 with $(\Gamma_0, G^{\Gamma_0}) \in \mathcal{Q}$, $G \cong G^{\Gamma_0}$, and G is faithful but not quasiprimitive on the excluded subset of elements.

We prove a little more for some of these cases in Section 5. In particular, if $k \geq 4$ in case (iii), then m is a prime power, and the upper bound $k = m + 1$ can be achieved for each such m , see Example 5.8. We note that, for a G -normal quotient of a G -incidence transitive pregeometry, incidence in the original pregeometry has a certain uniformity relative to the quotient: for each ordered pair (i, j) of types, there is a constant k_{ij} such that, whenever parts B_i, B_j (consisting of type i and type j elements respectively) are incident in the quotient, each element of B_i is incident with exactly k_{ij} elements of B_j (see [5, Lemma 6.6]). For some applications this uniformity is important, and it does not hold for G -imprimitive quotients in general. For such applications only Theorem 1.2 can be used.

For a pregeometry to be a geometry each flag must be contained in a chamber (that is, a flag containing an element of each type). If $(\Gamma, G) \in \mathcal{G}$ with Γ a geometry, then in general neither an imprimitive quotient nor a normal quotient of Γ can be guaranteed to be a geometry, see [5]. Several sufficient conditions were obtained in [5] for quotients of geometries to be geometries. In a forthcoming paper [12] we will pursue the study of G -primitive basic geometries Γ , showing that, for each O’Nan–Scott type of primitive group G , there are examples of thick geometries Γ of unboundedly large rank.

1.1. Outline of the paper. Section 2 covers preliminary theory of pregeometries including imprimitive and normal quotients, and also some results on permutation group actions. In Section 3.2 we prove Theorem 1.1, and in Section 4 we give a number of results concerning group actions on normal-basic pregeometries. We then use these results to prove Proposition 4.3 which lists the ways in which a group G can act on a normal-basic pregeometry (and which proves most of Theorem 1.2). In Section 5 we deal explicitly with case (iii) of Proposition 4.3 (namely when the incidence graph

of the pregeometry is complete multipartite), giving detailed information about the automorphism group actions in this case. This then enables us to prove Theorem 1.2.

2. PRELIMINARIES

2.1. Incidence pregeometries. By a *pregeometry* $\Gamma = (X, *, t)$ we mean a set X of *elements* (often called *points*) equipped with an *incidence relation* $*$ on the points, and a map t from X onto a set I of *types*. The incidence relation is symmetric and reflexive, and if $x * y$ we say that x and y are *incident*. Furthermore if $x * y$ with $x \neq y$ then $t(x) \neq t(y)$. For each $i \in I$, we write X_i to denote $t^{-1}(i)$, the set of all elements of type i , and so X is the disjoint union $\bigcup_{i \in I} X_i$. The number of types $|I|$ is called the *rank* of the pregeometry. We assume throughout the paper that $|X|$, and hence $|I|$, is finite. Unless stated otherwise, the set I for a rank k pregeometry is equal to $\{1, \dots, k\}$.

A typical example is a projective space: the elements are the non-trivial proper subspaces of a vector space, two subspaces are incident if one is contained in the other, and given a subspace α , the type $t(\alpha)$ is the algebraic dimension of α .

For a pregeometry $\Gamma = (X, *, t)$, and a nonempty subset J of the type set I , the J -truncation of Γ is the pregeometry $(X_J, *_J, t_J)$ where $X_J = t^{-1}(J)$, $*_J$ is the restriction of $*$ to $X_J \times X_J$, and t_J is the restriction of t to X_J .

The *incidence graph* of Γ is the graph with vertex set X , and edges $\{x_i, x_j\}$ whenever $x_i * x_j$ and $i \neq j$, where x_i has type i and x_j has type j . Since different elements of the same type are never incident, such a graph is always multipartite. If the incidence graph is complete multipartite (that is, $\{x_i, x_j\}$ is an edge whenever $i \neq j$), then we say that the pregeometry Γ is itself complete multipartite. Furthermore, we use the notation $\Gamma(k, m)$ to mean the rank k complete multipartite pregeometry in which $|X_i| = m$ for all i .

Throughout the paper we study pregeometries Γ in which the rank 2 truncations are connected. This means that for each pair of distinct types i, j , the incidence graph of the $\{i, j\}$ -truncation of Γ is connected.

Let $\Gamma = (X, *, t)$ be an incidence pregeometry. An *automorphism* of Γ is a permutation g of X such that $t(x) = t(x^g)$ for all $x \in X$, and $x^g * y^g$ if and only if $x * y$, for all $x, y \in X$. We write $\text{Aut } \Gamma$ for the automorphism group of Γ . As in Section 1, we write \mathcal{G} for the set of all pairs (Γ, G) where Γ is a pregeometry with all rank 2 truncations connected, and $G \leq \text{Aut}(\Gamma)$ is transitive on each X_i .

2.2. Permutation groups. In this section we introduce some notation and results pertaining to group actions. Let G be a group acting on a set Y . We write $G_{(Y)}$ for the kernel of the G -action on Y , and G^Y for the group induced by G on Y (isomorphic to $G/G_{(Y)}$). The action of G on Y is called *faithful* if $G_{(Y)} = 1$, so that $G^Y \cong G$. For $y \in Y$, we write y^G for the orbit of y under G . The action of G on Y is called *semi-regular* if for all $y \in Y$, the stabiliser $G_y \leq G_{(Y)}$. It is called *regular* if it is both semi-regular and transitive.

For a subgroup T of G , the *centraliser* $C_G(T)$ of T in G is the set of all $g \in G$ which commute with every $t \in T$. The following result is taken from [6, Theorem 4.2A].

Lemma 2.1 (Centraliser Lemma). *Let T be a transitive subgroup of a permutation group G on Y , and let $S \leq C_G(T)$. Then S is semi-regular on Y . Moreover, if S is transitive on Y , then S is regular, $S = C_G(T)$, $T \cong S$, and T is also regular on Y . If in addition T is abelian, then $T = S$.*

Our next result is a generalisation of [7, Lemma 5.4], which dealt with the case where $k = 2$.

Lemma 2.2. *Suppose that G acts on a set X with orbits X_1, X_2, \dots, X_k such that, for all nontrivial normal subgroups N of G , N is transitive on all but at most one G -orbit. Then G is quasiprimitive on all but at most one G -faithful orbit.*

Proof. Suppose that there exist distinct i, j such that G is faithful but not quasiprimitive on X_i and X_j . Then there exist nontrivial normal subgroups N_i, N_j of G such that $N_i^{X_i}$ and $N_j^{X_j}$ are intransitive and nontrivial. Since each of N_i and N_j is transitive on all but at most one G -orbit, it follows that $N_i \neq N_j$ and $N_i^{X_j}, N_j^{X_i}$ are transitive. Now $N_i \cap N_j \triangleleft G$ and is intransitive on both X_i and X_j . Hence $N_i \cap N_j = 1$ and so $N_j \leq C_G(N_i)$ and $N_i \leq C_G(N_j)$. By the faithfulness of G on X_i, X_j and by Lemma 2.1, we have that N_j acts semiregularly and intransitively on X_j and N_i acts semiregularly and intransitively on X_i . Thus $|N_j| < |X_j|$ and $|N_i| < |X_i|$. Moreover, by transitivity we have that $|X_j|$ divides $|N_i|$ and $|X_i|$ divides $|N_j|$. Thus $|N_j| < |X_j| \leq |N_i| < |X_i| \leq |N_j|$, which is a contradiction. It follows that G is quasiprimitive on all but at most one G -faithful X_i . \square

2.3. Quotients of incidence pregeometries. Let $\Gamma = (X, *, t)$ be a pregeometry and let \mathcal{P} be a type-refining partition of X ; that is, each part of \mathcal{P} is contained in one of the sets X_i . Then the *quotient* of Γ by \mathcal{P} is the pregeometry $\Gamma_{/\mathcal{P}} = (\mathcal{P}, */_{\mathcal{P}}, t_{/\mathcal{P}})$ where for $P, P' \in \mathcal{P}$ we have $P */_{\mathcal{P}} P'$ if and only if there exists $x \in P$ and $y \in P'$ with $x * y$; and $t_{/\mathcal{P}}(P) = t(x)$ for $x \in P$. We say that $\Gamma_{/\mathcal{P}}$ is a proper quotient if $|\mathcal{P}| < |X|$.

Let $G \leq \text{Aut } \Gamma$. A partition \mathcal{P} of X is G -invariant if G permutes the parts of \mathcal{P} setwise. If \mathcal{P} is a type-refining G -invariant partition of X then $\Gamma_{/\mathcal{P}}$ is called a G -*imprimitive* quotient of Γ , or simply an imprimitive quotient if the group is clear from the context.

Let N be a normal subgroup of G , and let \mathcal{P}_N be the set of N -orbits on X . Then \mathcal{P}_N is a G -invariant partition of X , and the G -*normal quotient*, or simply normal quotient, of Γ by N is the pregeometry $\Gamma_{/\mathcal{P}_N}$. We usually abbreviate the notation to $\Gamma_{/N}$.

As mentioned in the introduction, the motivation for using quotients to study pregeometries is that it gives a framework for characterising pregeometries as being built up from ‘basic’ examples (that is, examples which admit no ‘non-degenerate’

quotients). As the terms ‘degenerate’ and ‘basic’ have slightly different meaning in the contexts of imprimitive and normal quotients, we give their definitions separately in Sections 3.1 and 4.1.

2.4. Decomposition of pregeometries. When characterising imprimitive-basic and normal-basic pregeometries we use the notion of indecomposability. For this we use the direct sum of pregeometries (as it appears in [2, p 80]).

Construction 2.3. Let $\Gamma_i = (X^{(i)}, *^{(i)}, t^{(i)})$ with type set $I^{(i)}$ for $i = 1, 2$. The *direct sum* of Γ_1 and Γ_2 is

$$\Gamma_1 \oplus \Gamma_2 = (X, *, t)$$

where $X = X^{(1)} \dot{\cup} X^{(2)}$, $I = I^{(1)} \dot{\cup} I^{(2)}$, t induces $t^{(i)}$ on $X^{(i)}$, $*$ induces $*^{(i)}$ on the restriction to Γ_i , and $x^{(1)} * x^{(2)}$ for all $x^{(1)} \in X^{(1)}$ and $x^{(2)} \in X^{(2)}$.

Note that $\text{Aut}(\Gamma_1 \oplus \Gamma_2)$ is equal to the direct product $\text{Aut} \Gamma_1 \times \text{Aut} \Gamma_2$.

We say that a pregeometry Γ is *decomposable* if there exist Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \oplus \Gamma_2$. If Γ is not decomposable then it is called *indecomposable*.

3. PRIMITIVE-DEGENERATE AND PRIMITIVE-BASIC PREGEOMETRIES

Throughout this section $\Gamma = (X, *, t)$ denotes a pregeometry with type set I and $G \leq \text{Aut} \Gamma$. For $J \subseteq I$, we say that the J -truncation Γ_J is *fully G -faithful*, or *fully G -primitive* if the group G^{Γ_J} induced by G on Γ_J is faithful on X_j for each $j \in J$, or primitive on X_j for each $j \in J$, respectively.

3.1. Definitions. We use the term *effective rank* of a pregeometry to mean the number of types which have more than one element. For our theory of imprimitive quotients we use proper quotients of a pregeometry having the same effective rank as the original pregeometry, or equivalently, no part in the partition should be equal to an entire set X_i in the original pregeometry.

Provided G leaves invariant a proper, non-trivial partition \mathcal{P}_i of at least one of the X_i , we can always extend \mathcal{P}_i to a partition \mathcal{P} of X such that \mathcal{P} partitions each of the remaining X_j into singleton subsets (irrespective of the G -action on X_j), giving a proper quotient pregeometry with the same effective rank.

We say that a pregeometry Γ is *primitive-degenerate* if at least one of the X_i contains only one element. The corresponding notion of basic, for a pregeometry Γ with $G \leq \text{Aut} \Gamma$, is as follows: Γ is called *G -primitive-basic* if Γ is not primitive-degenerate but every proper G -imprimitive quotient is primitive-degenerate. The latter condition is equivalent to requiring that Γ is fully G -primitive.

3.2. Proof of Theorem 1.1. We begin with the following lemma.

Lemma 3.1. *Suppose that $\Gamma = (X, *, t)$ is a pregeometry of rank at least two. Suppose also that $G \leq \text{Aut} \Gamma$, G is transitive on each X_i , and $1 \neq N \triangleleft G$ such that N acts trivially on X_1 and transitively on X_j for some $j \neq 1$. Assume that there exist $x_1 \in X_1$*

and $x_j \in X_j$ with $x_1 * x_j$. Then each element of X_1 is incident with each element of X_j .

Proof. Let $x_1 \in X_1$ and $x_j \in X_j$ such that $x_1 * x_j$. Since $x_1^G = X_1$, $x_1^N = \{x_1\}$, $x_j^N = X_j$ and G preserves incidence, the result follows. \square

Lemma 3.2. *Let $(\Gamma, G) \in \mathcal{G}$ and suppose that Γ is fully G -primitive. If Γ is indecomposable then Γ is fully G -faithful.*

Proof. Assume that G is unfaithful on some set X_i . Let $T_i = G_{(X_i)}$, and let I_i be the set of all j with $T_i \leq G_{(X_j)}$. Let $I' = I \setminus I_i$. Since G is faithful on X , $|I'| \geq 1$. Let Γ_i be the I_i -truncation and Γ' the I' -truncation of Γ . Let $j \in I'$ and let $x \in \bigcup_{\ell \in I_i} X_\ell$. Since the rank 2 truncations are connected there exists $x_j \in X_j$ with $x * x_j$. Since G is primitive on X_j , the non-trivial normal subgroup T_i of G is transitive on X_j , while fixing the point x in $\bigcup_{\ell \in I_i} X_\ell$. Thus by Lemma 3.1, $x * x'_j$ for all $x'_j \in X_j$. As this holds for all $x \in \bigcup_{\ell \in I_i} X_\ell$, it follows that $\Gamma = \Gamma_i \oplus \Gamma'$, and so Γ is decomposable. \square

Corollary 3.3. *Let $(\Gamma, G) \in \mathcal{G}$ and suppose that Γ is fully G -primitive. Then there exists a unique partition $\mathcal{I} = \{I_1, \dots, I_\ell\}$ of I such that, for each i , the truncation Γ_{I_i} is an indecomposable, fully G -faithful, fully G -primitive pregeometry, and $\Gamma = \Gamma_{I_1} \oplus \dots \oplus \Gamma_{I_\ell}$.*

Proof. The existence of such a partition follows from Lemma 3.2, Lemma 3.1 and the fact that all rank 2 truncations are connected. Suppose that $\Gamma = \Gamma_{J_1} \oplus \dots \oplus \Gamma_{J_r}$ is another decomposition and let $n \in \{1, \dots, \ell\}$. Now $I_n = (I_n \cap J_1) \dot{\cup} (I_n \cap J_2) \dot{\cup} \dots \dot{\cup} (I_n \cap J_r)$. For $i = 1, 2, \dots, r$, let Γ_i be the $(I_n \cap J_i)$ -truncation of Γ_{I_n} . Then $\Gamma_{I_n} = \bigoplus_{j \in J} \Gamma_{I_j}$, where $J = \{j \mid 1 \leq j \leq r, I_n \cap J_j \neq \emptyset\}$. Since Γ_{I_n} is indecomposable, it follows that $|J| = 1$ so there exists $j \in \{1, \dots, r\}$ such that $I_n \subseteq J_j$. Applying the same argument to J_j implies that $I_n = J_j$. Hence the partition \mathcal{I} is unique. \square

Theorem 1.1 follows from the above Corollary. The following example illustrates a decomposable example in which each of the Γ_i is an “interesting” geometry.

Example 3.4. Let Γ be the geometry whose points are the elements of the projective space $\text{PG}(d-1, q)$ and incidence is the usual incidence (that is, subspace inclusion) with the added incidence that every 1-space and 2-space is incident with every subspace of dimension at least 3. Let i be the type assigned to the i -spaces; so X_i is the set of all i -spaces.

Let $G = \text{PSL}(d, q) \times \text{PSL}(d, q)$ act on this geometry where the first factor acts non-trivially on X_1 and X_2 and trivially on the rest, while the second factor acts trivially on X_1 and X_2 but non-trivially on the rest. Since G acts primitively on each type, Γ is G -primitive-basic.

The group G acts unfaithfully on each of the X_i . However, the incidence graph of Γ is not complete multipartite, as $\Gamma = \Gamma_1 \oplus \Gamma_2$ where Γ_1 and Γ_2 are, respectively, the $\{1, 2\}$ -truncation and $\{3, \dots, d-1\}$ -truncation of Γ , and each of Γ_1 and Γ_2 is indecomposable, with each being a truncation of $\text{PG}(d-1, q)$.

4. NORMAL-DEGENERATE AND NORMAL-BASIC PREGEOMETRIES

4.1. Normal degeneracy. Let $(\Gamma, G) \in \mathcal{G}$. In the case of normal quotients, we do not have the freedom to choose the induced partitions of the X_i independently of one another, as they are all determined by the orbits of a given normal subgroup $N \trianglelefteq G$. Thus it is less reasonable (and less useful) to use the notion of primitive-degeneracy in this context. Instead we require for non-degeneracy only that the quotient pregeometry has effective rank at least two (that is, $|X_i| \geq 2$ for at least two types i).

A pregeometry Γ is called *normal-degenerate* if at most one of the X_i contains more than one element. Corresponding to this notion, a pregeometry Γ with $G \leq \text{Aut } \Gamma$ is called *G-normal-basic* if Γ is not normal-degenerate, and every proper quotient is normal-degenerate. The latter condition is equivalent to requiring that every non-trivial normal subgroup of G is transitive on all but at most one of the X_i .

4.2. Faithful, unfaithful and quasiprimitive orbits. For a permutation group G on a set X we say that a G -orbit is *G-faithful* if G acts faithfully on it, and otherwise it is called *G-unfaithful*. Let $\Gamma = (X, *, t)$ be a pregeometry with $G \leq \text{Aut } \Gamma$. A type $i \in I$ is called *G-unfaithful* or *G-faithful* according to whether $G_{(X_i)} \neq 1$ or $G_{(X_i)} = 1$ respectively, and i is called *G-quasiprimitive* if G^{X_i} is quasiprimitive. If every $i \in I$ is G -quasiprimitive and G -faithful then Γ is called *fully G-quasiprimitive*.

Lemma 4.1. *Suppose that $(\Gamma, G) \in \mathcal{G}$ has rank at least two and for all nontrivial normal subgroups N of G , N is transitive on all but at most one of the X_i . Then either Γ is complete multipartite or G is faithful on at least two of the X_i .*

Proof. We have $X = X_1 \cup \dots \cup X_k$. Let J be the subset of I consisting of all j such that G is unfaithful on X_j . If $|I \setminus J| \geq 2$ then the second conclusion follows. Hence assume that $|I \setminus J| \leq 1$ and let $j \in J$. Then $G_{(X_j)} \neq 1$. Let $i \in I \setminus \{j\}$. By assumption, $G_{(X_j)}$ is transitive on X_i . By Lemma 3.1, every element of type j is incident with every element of type i . Since this holds for all $j \in J$ and all $i \neq j$, it follows that the incidence graph of Γ is complete multipartite. \square

Define I_{unf} to be the set of all G -unfaithful types; I_{qp} the set of all types that are both G -faithful and G -quasiprimitive; and I_{nonqp} the set $I \setminus (I_{\text{unf}} \cup I_{\text{qp}})$. In particular, we have a partition of I into the three disjoint parts I_{qp} , I_{unf} and I_{nonqp} .

Lemma 4.2. *Let $(\Gamma, G) \in \mathcal{G}$ such that for all nontrivial normal subgroups N of G , N is transitive on all but at most one of the X_i . Then*

$$|I_{\text{unf}} \cup I_{\text{nonqp}}| \leq \# \text{ minimal normal subgroups of } G.$$

Moreover, if $|I_{\text{qp}}| \geq 1$ then $|I_{\text{unf}} \cup I_{\text{nonqp}}| \leq 2$ with equality implying $|I_{\text{unf}}| = 2$.

Proof. Let $j \in I_{\text{unf}} \cup I_{\text{nonqp}}$ and let m be the number of minimal normal subgroups of G . Then there exists a minimal normal subgroup N_j of G such that $N_j^{X_j}$ is intransitive.

By assumption, for all $i \neq j$, $N_j^{X_i}$ is transitive. Thus for distinct $j, j' \in I_{\text{unf}} \cup I_{\text{nonqp}}$, $N_j \neq N_{j'}$ and hence $|I_{\text{unf}} \cup I_{\text{nonqp}}| \leq m$. Moreover, if $|I_{\text{qp}}| \geq 1$ then by [4, Theorem 4.4], G has at most two minimal normal subgroups and so $|I_{\text{unf}} \cup I_{\text{nonqp}}| \leq 2$.

Suppose now that $|I_{\text{qp}}| \geq 1$ and $|I_{\text{unf}} \cup I_{\text{nonqp}}| = 2$. Then G has exactly two minimal normal subgroups N, M , say, and in particular, each type in I_{qp} is primitive. Moreover, $N \cong M \cong T^n$ for some finite non-abelian simple group T and positive integer n (see [6, Theorem 4.3B]). Also $N \cap M = 1$, so $\langle N, M \rangle = NM$. Let $i \in I_{\text{unf}} \cup I_{\text{nonqp}}$. Then without loss of generality we may assume that N^{X_i} is intransitive. Suppose that $j \in I_{\text{unf}} \cup I_{\text{nonqp}}$ with $j \neq i$. By assumption N^{X_j} is transitive and so, for $\alpha \in X_j$, $G = G_\alpha N$. Note that N^{X_j} being transitive implies that $|X_j|$ divides $|N|$ and that $(NM)^{X_j}$ is transitive. Let $H = G_\alpha \cap (NM)$ and let π_2 be the projection map from NM to M . Then $H \triangleleft G_\alpha$ and so HN is normalised by $G_\alpha N = G$. Now $HN \leq NM$ and $M \cong (NM)/N$ is a minimal normal subgroup of G/N . Hence $HN/N = MN/N$ and so $HN = MN$. Thus $M \cong H/(H \cap N)$ and so $\pi_2(H) = M$. Now $H \cap M = G_\alpha \cap (NM) \cap M = G_\alpha \cap M$, which is normalised by G_α . Since $G = G_\alpha N$ and N centralises M , G_α acts transitively on the set of simple direct factors of M . Moreover, as $H \cap M \triangleleft H$ we have $\pi_2(H \cap M) \triangleleft \pi_2(H) = M$. Thus $H \cap M = M$ or 1 . Since $(NM)^{X_j}$ is transitive, we have $|X_j| = |NM : H|$. If $H \cap M = 1$ then $M_\alpha = 1$ and $|NM : H| = |NM : HM||HM : H| = |NM : HM||M|$ and so $|M|$ divides $|NM : H| = |X_j|$. As noted above, $|X_j|$ divides $|N| = |M|$, so $|M : M_\alpha| = |M| = |X_j|$ and M^{X_j} is transitive. Thus each minimal normal subgroup of G is transitive on X_j and hence G is quasiprimitive on X_j , which is a contradiction. Thus $H \cap M = M$, and in particular M , a normal subgroup of G is contained in G_α . Hence G acts unfaithfully on X_j and so $j \in I_{\text{unf}}$. Moreover, applying the same argument with i and j interchanged we find that N acts trivially on X_i , and $i \in I_{\text{unf}}$. Thus $|I_{\text{unf}} \cup I_{\text{nonqp}}| = 2$, which implies $I_{\text{nonqp}} = \emptyset$. \square

4.3. Group actions on normal-basic pregeometries. Let $\Gamma = (X, *, t)$ be a pregeometry and $G \leq \text{Aut } \Gamma$. Recall that if G has a faithful quasiprimitive action on each X_i then we say that Γ is *fully G -quasiprimitive*. Let \mathcal{Q} denote the set of all pairs (Γ, G) where Γ is a pregeometry and $G \leq \text{Aut } \Gamma$ such that Γ is fully G -quasiprimitive. Recall also that we use the notation $\Gamma(k, m)$ to denote the rank k complete multipartite pregeometry in which $|X_i| = m$ for all i .

Our next Proposition lists all the possible ways a group G can act on a normal-basic pregeometry.

Proposition 4.3. *Let $(\Gamma, G) \in \mathcal{G}$ such that Γ is a G -normal basic pregeometry of rank k . Then exactly one of the following holds.*

- (i) $(\Gamma, G) \in \mathcal{Q}$.
- (ii) (a) $k \geq 3$ and $\Gamma = \Gamma_0 \oplus \Gamma(1, m)$ where $(\Gamma_0, G^{\Gamma_0}) \in \mathcal{Q}$, $G \cong G^{\Gamma_0}$, and the single type of $\Gamma(1, m)$ is G -unfaithful.

- (b) $k \geq 4$ and $\Gamma = \Gamma_0 \oplus \Gamma(1, m) \oplus \Gamma(1, m')$ where $(\Gamma_0, G^{\Gamma_0}) \in \mathcal{Q}$, $G \cong G^{\Gamma_0}$, and G is unfaithful on the point set of each of $\Gamma(1, m)$ and $\Gamma(1, m')$. In this case G has 2 minimal normal subgroups.
- (iii) Γ is complete multipartite and G is faithful on at most one of the X_i .
- (iv) G is faithful but not quasiprimitive on one of the X_i , and faithful and quasiprimitive on each of the others.

Proof. Let $s = |I_{\text{unf}} \cup I_{\text{nonqp}}|$ (as defined before Lemma 4.2). If $s = 0$ then Γ is fully quasiprimitive as in case (i) of the statement. Suppose next that $s = 1$. If $|I_{\text{nonqp}}| = s = 1$ then we are in case (iv). If $|I_{\text{unf}}| = s = 1$ and $k = 2$ then we are in case (iii) by Lemma 3.1. Assume next that $|I_{\text{unf}}| = s = 1$ and $k \geq 3$. Let $I_{\text{unf}} = \{j\}$ and let Γ_0 be the $(I \setminus \{j\})$ -truncation and Γ' the $\{j\}$ -truncation of Γ . Then $\Gamma' = \Gamma(1, m)$ where $m = |X_j|$. Since Γ is G -normal-basic, so is Γ_0 , and as G is faithful and quasiprimitive on all X_i in Γ except X_j , Γ_0 is fully G -quasiprimitive. That $\Gamma = \Gamma_0 \oplus \Gamma(1, m)$ is an immediate consequence of Lemma 3.1. Hence we are in case (ii)(a).

Suppose now that $s = 2$. Then by Lemma 4.2, G has (at least) 2 minimal normal subgroups. Moreover, by Lemma 2.2, $|I_{\text{nonqp}}| \leq 1$. Thus $|I_{\text{unf}}| \geq 1$. If $k = 2$ then we are in case (iii) by Lemma 3.1. On the other hand if $k \geq 3$ then $|I_{\text{qp}}| = k - s \geq 1$ and so G has at most two minimal normal subgroups (see [13, Theorem 1]). Since $s = 2$, Lemma 4.2 implies that there are exactly two minimal normal subgroups. Also, Lemma 4.2 implies that $|I_{\text{unf}}| = 2$. By Lemma 4.1, either $k = 3$ and we are in case (iii), or $k \geq 4$. Assume that $k \geq 4$, let $I_{\text{unf}} = \{j, \ell\}$, let Γ_0 be the $(I \setminus \{j, \ell\})$ -truncation and Γ' the $\{j, \ell\}$ -truncation of Γ . Let $m = |X_j|$ and $m' = |X_\ell|$. By assumption $G_{(X_j)}$ is transitive on X_ℓ , so by Lemma 3.1, every point in X_j is incident with every point in X_ℓ , and hence $\Gamma' = \Gamma(1, m) \oplus \Gamma(1, m')$. Since Γ is G -normal-basic, so is Γ_0 , and as G is faithful and quasiprimitive on each X_i with $i \in I \setminus \{j, \ell\}$, Γ_0 is fully G -quasiprimitive. That $\Gamma = \Gamma_0 \oplus \Gamma'$ also follows from Lemma 3.1 (applied with $X_1 = X_j$ and then again with $X_1 = X_\ell$ in its statement). Hence we are in case (ii)(b).

Suppose finally that $s > 2$. Then by Lemma 4.2, $|I_{\text{qp}}| = \emptyset$ and so $s = k$. Moreover, Lemma 2.2 implies that $|I_{\text{unf}}| \geq k - 1$ and so by Lemma 4.1, Γ is complete multipartite. Hence we are in case (iii) and the proof is complete. \square

5. CASE (III) OF PROPOSITION 4.3

In Proposition 5.3 we give further details (shown in Table 1) about G -normal-basic pregeometries in case (iii) of Proposition 4.3. We work with the following hypothesis concerning Γ , G and k .

Hypothesis 5.1. *Suppose that $\Gamma = (X, *, t)$ is a G -normal-basic pregeometry of rank $k \geq 2$, such that Γ is in case (iii) of Proposition 4.3, that is, Γ is complete multipartite, and G is faithful on at most one of the X_i . For each type i , let $T_i = G_{(X_i)}$.*

First we show via construction that this case arises for rank 2 with $|I_{\text{unf}}| = 2$ and any $|X_1|, |X_2|$.

Construction 5.2. Let X_1 and X_2 be sets and for each i let T_i be a group acting faithfully and quasiprimively on X_i . Let Γ be the pregeometry with point set $X_1 \dot{\cup} X_2$ such that the incidence graph is complete bipartite. Let $G = T_1 \times T_2$ and let G act on Γ by

$$x_i^{(t_1, t_2)} = x_i^{t_i}$$

if $x_i \in X_i$. Clearly $G \leq \text{Aut } \Gamma$ and Γ is G -normal basic. Furthermore, G acts unfaithfully on each X_i .

Our main result for $k \geq 3$ is Proposition 5.3. We use the notation I_{unf} and I_{qp} as defined in Section 4.2.

Proposition 5.3. *Suppose that Γ and G satisfy Hypothesis 5.1 with $k \geq 3$. Then each T_i is a minimal normal subgroup, and k , $|I_{\text{unf}}|$, $|I_{\text{qp}}|$ and the T_i are as in one of the lines of Table 1. Moreover, if $|I_{\text{unf}}| \geq 3$ then $\Gamma = \Gamma(k, m)$ for some m .*

	k	$ I_{\text{unf}} $	$ I_{\text{qp}} $	Comments on T_i
(i)	3	2	1	nonabelian, $T_1 \cong T_2$, $ X_3 = T_1 $, $ X_1 $ and $ X_2 $ divide $ X_3 $
(ii)	3	3	0	nonabelian, $\langle T_1, T_2, T_3 \rangle = T_1 \times T_2 \times T_3$
(iii)	$3 \leq k \leq m+1$	k	0	$T_i = \mathbb{Z}_p^d$, $m = p^d$, p a prime, $\langle T_i \mid i \in I \rangle = \mathbb{Z}_p^{2d}$

TABLE 1. The possibilities in Proposition 5.3.

We prove the above Proposition using Lemmas 5.4 and 5.6–5.10, and Corollary 5.5 below.

Lemma 5.4. *Suppose that Γ and G satisfy Hypothesis 5.1 with $k \geq 3$. For distinct $i, j \in I_{\text{unf}}$ and $\ell \in I \setminus \{i, j\}$, the following hold.*

- (a) $T_i \cap T_j = 1$,
- (b) T_i and T_j are faithful and regular on X_ℓ , and
- (c) $T_i \cong T_j$.

Proof. For any distinct $i', j' \in I$, the subgroups $T_{i'}$ and $T_{j'}$ are normal in G , so $T_{i'} \cap T_{j'} \triangleleft G$ and $T_{i'} \cap T_{j'}$ acts trivially on both $X_{i'}$ and $X_{j'}$. Since Γ is G -basic, this implies that $T_{i'} \cap T_{j'} = 1$. In particular part (a) is proved, and T_i, T_j are both faithful on X_ℓ .

Now, since Γ is G -basic, $T_i^{X_\ell}$ and $T_j^{X_\ell}$ are both transitive. Since $T_i \cap T_j$ is trivial and T_i and T_j normalise each other, we have $T_i \leq C_G(T_j)$. Hence by Lemma 2.1, T_i and T_j are regular on X_ℓ , and $T_i \cong T_j$, whence we get parts (b) and (c). \square

Corollary 5.5. *Suppose that Γ and G satisfy Hypothesis 5.1, and $|I_{\text{unf}}| \geq 3$. Then $\Gamma = \Gamma(k, m)$ for some m .*

Proof. Let $i, j \in I_{\text{unf}}$ with $i \neq j$, let $\ell \in I \setminus \{i, j\}$, and let $i' \in I_{\text{unf}} \setminus \{i, j\}$. By Lemma 5.4, T_i is regular and faithful on X_ℓ . The same argument with $\{i, i'\}$, j in place of $\{i, j\}$, ℓ gives that T_i is regular and faithful on X_j . So $|T_i| = |X_j| = |X_\ell|$. Similarly $|X_i| = |X_\ell|$. Writing $m := |X_\ell|$ we obtain the result. \square

Lemma 5.6. *Suppose that Γ and G satisfy Hypothesis 5.1 such that $k = 3$, and $|I_{\text{unf}}| = 2$, say $I_{\text{unf}} = \{1, 2\}$. Then G is quasiprimitive on X_3 and T_1, T_2 are its minimal normal subgroups. Moreover, $\Gamma = \Gamma(1, m_1) \oplus \Gamma(1, m_2) \oplus \Gamma(1, m_3)$ where $m_3 = |T_1| = |T_2|$ and m_1, m_2 divide m_3 .*

Proof. The fact that Γ is G -basic implies that $T_i^{X_3} \cong T_i$ is transitive for $i = 1, 2$. By Lemma 5.4, $T_1 \cap T_2 = 1$ so $T_1 \leq C_G(T_2)$ and hence by Lemma 2.1, $T_1 \cong T_2$ and both are faithful and regular on X_3 . Hence $|T_1| = |T_2| = |X_3| = m_3$. Moreover, $T_1^{X_2} \cong T_1$, $T_2^{X_1} \cong T_2$ are both transitive, and so, for $i \in \{1, 2\}$, $m_i = |X_i|$ divides $m_3 = |T_{3-i}|$. Thus, since Γ is complete multipartite by Hypothesis 5.1, Γ is as stated. Also since $G \cong G^{X_3}$ and $T_1 \cap T_2 = 1$, it follows by Lemma 2.1 that $T_1 = C_G(T_2)$.

Now assume that $N \trianglelefteq G$ with N intransitive on X_3 . Then for $i = 1, 2$, $N \cap T_i \triangleleft G$ and $N \cap T_i$ is intransitive on both X_i and X_3 , so since Γ is G -basic, $N \cap T_i$ is trivial. Hence $N \leq C_G(T_i) = T_{3-i}$, so $N \leq T_1 \cap T_2 = 1$. Thus all nontrivial normal subgroups of G are transitive on X_3 , that is, G^{X_3} is quasiprimitive. Since $T_1^{X_3}$ and $T_2^{X_3}$ are regular, it follows that T_1, T_2 are the minimal normal subgroups of G . \square

For isomorphic groups A, B a full diagonal subgroup of $A \times B$ is a subgroup $C \leq A \times B$ such that C projects onto both A and B , and for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in C$. Each full diagonal subgroup is of the form $C = \{(a, a^\varphi) \mid a \in A\}$ for some isomorphism $\varphi : A \rightarrow B$.

Lemma 5.7. *Suppose that Γ and G satisfy Hypothesis 5.1 with $k \geq 3$ and $|I_{\text{unf}}| \geq 3$, and that $T_i = G_{(X_i)}$ is abelian for some type i . Then for some prime p and integer d , $\Gamma = \Gamma(k, p^d)$, where $k \leq p^d + 1$, each $T_i \cong \mathbb{Z}_p^d$, is a minimal normal subgroup of G , and $\langle T_i \mid i \in I \rangle = \mathbb{Z}_p^{2d}$.*

Proof. First we note that, by Lemma 5.4 (c), $T_i \cong T_u$ for all types u , so T_u is abelian also.

Let i and j be distinct types, and let N be a minimal normal subgroup of G contained in T_i . Now T_i is abelian, so $N = \mathbb{Z}_p^d$ for some prime p . Moreover, by Lemma 5.4 (b), T_i acts faithfully and regularly on X_j . Thus N is faithful and semi-regular on X_j and so as Γ is G -basic, $N = T_i$ so T_i is a minimal normal subgroup of G , and $|N| = |X_j|$. Since there are at least 3 unfaithful types, repeating this argument we have $|X_\ell| = p^d$ for all ℓ . It follows from the fact that Γ is complete multipartite that $\Gamma = \Gamma(k, m)$ where $m = p^d$.

Again, let i and j be distinct types, and note that $T_i \cong T_j \cong \mathbb{Z}_p^d$ and $T_i \times T_j \leq G$ by Lemma 5.4.

Claim: For each type $\ell \neq i, j$, T_ℓ is a full diagonal subgroup of $T_i \times T_j$ and $\langle T_r \mid r \in I \rangle = T_i \times T_j \cong \mathbb{Z}_p^{2d}$.

Proof of claim: Since T_i and T_j centralise each other, the groups $T_i^{X_\ell}$ and $T_j^{X_\ell}$ centralise each other as subgroups of $\text{Sym}(X_\ell)$, and since each of these subgroups is abelian Lemma 2.1 implies that $T_i^{X_\ell} = T_j^{X_\ell}$.

Thus, let $S_\ell := \{t_i t_j \mid t_i \in T_i, t_j \in T_j, t_j^{X_\ell} = (t_i^{X_\ell})^{-1}\}$. Since $[T_i, T_j] = 1$ the set S_ℓ is a subgroup of $T_i \times T_j$. In fact it is a full diagonal subgroup of $T_i \times T_j \leq G$, implying that $S_\ell \cong T_i \cong T_j$, and also S_ℓ acts trivially on X_ℓ . It follows from Lemma 5.4 (c) that $S_\ell = T_\ell$. Hence the claim is proved.

Without loss of generality we let $\{i, j\} = \{1, 2\}$. Identifying each of T_1 and T_2 with the vector space $V' := V(d, p)$ we may identify the group induced by the conjugation action of G on $T_1 \times T_2$ with a subgroup \hat{G} of $\text{GL}(2d, p)$ in its action on $V := V' \oplus V'$. Under this identification, the group T_ℓ corresponds to a subspace $V_A := \{(u, uA) \mid u \in V'\}$ for some matrix A in $\text{GL}(d, p)$.

Let $g \in G$. As G normalises each of T_1 and T_2 , there exist B_1 and B_2 in $\text{GL}(d, p)$ such that the element \hat{g} of \hat{G} induced by g is equal to

$$\left(\begin{array}{c|c} B_1 & \mathbf{0} \\ \hline \mathbf{0} & B_2 \end{array} \right)$$

where $\mathbf{0}$ denotes the $d \times d$ zero-matrix. Since $T_\ell \trianglelefteq G$, we have $V_A^{\hat{g}} = \{(uB_1, uAB_2) \mid u \in V'\} = V_A$, and this implies that $AB_2 = B_1A$, and hence $B_2 = A^{-1}B_1A$. Let $H \leq \text{GL}(d, p)$ be the image of \hat{G} under the projection

$$\left(\begin{array}{c|c} B_1 & \mathbf{0} \\ \hline \mathbf{0} & B_2 \end{array} \right) \mapsto B_1.$$

Now, let $s \in I \setminus \{1, 2, \ell\}$. By the Claim, T_s corresponds to a subspace $V_{A'} = \{(u, uA') \mid u \in V'\}$ for some $A' \in \text{GL}(d, p)$. Arguing as above we find that $B_2 = (A')^{-1}B_1A'$, and hence that $A^{-1}B_1A = (A')^{-1}B_1A'$. Thus, for all $B \in H$, $B = A'A^{-1}BA(A')^{-1}$ and so $A'A^{-1}$ centralises H . As T_1 is a minimal normal subgroup of G , it follows that H acts irreducibly on V' . By Schur's Lemma (see [1, (12,4)]), the centraliser of H in the matrix algebra $M(d, p)$ is a finite division ring and hence $C_{\text{GL}(d, p)}(H) \cong \text{GL}(1, p^e)$ for some e dividing d . Thus $|I \setminus \{1, 2\}| \leq p^e - 1 \leq p^d - 1$. \square

We give an example to show that the bound $k = p^d + 1$ can be achieved.

Example 5.8. Let $T = \mathbb{Z}_p^d$ for some prime p and integer d . Identify T with the additive group of the finite field \mathbb{F}_{p^d} of order p^d , and note that $\mathbb{F}_{p^d}^*$ acts transitively by field multiplication on T^* . Let $G = (T \times T).H$ where $H = \{(h, h) \mid h \in \mathbb{F}_{p^d}^*\}$. Then $T \times 1$ and $1 \times T$ are minimal normal subgroups of G . For all $\lambda \in \mathbb{F}_{p^d}$ define $T_\lambda = \{(u, \lambda u) \mid u \in T\}$. Also, define $T_\infty = 1 \times T$. Let $\{0, \infty\} \subseteq \Lambda \subseteq \mathbb{F}_{p^d}$. For each

$\lambda \in \Lambda$, define $X_\lambda = [G : T_\lambda.H]$, the set of right cosets of $T_\lambda.H$ in G , with G acting on X_λ by right multiplication. Thus $G_{(X_\lambda)} = T_\lambda$.

Define the pregeometry $\Gamma_\Lambda = (X, *, t)$ by $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, $t(x) = \lambda$ whenever $x \in X_\lambda$, and $x*y$ if either $x = y$ or $t(x) \neq t(y)$. It is straightforward to verify that $\Gamma_\Lambda \cong \Gamma(k, m)$ where $m = p^d$ and $k = |\Lambda| \leq p^d + 1$ (and if $\Lambda = \mathbb{F}_p^d \cup \{\infty\}$ then $k = p^d + 1$), and that Γ, G satisfy Hypothesis 5.1.

Lemma 5.9. *Suppose that Γ and G satisfy Hypothesis 5.1 with $k \geq 4$. Let i be an unfaithful type. Then T_i is abelian.*

Proof. By Hypothesis 5.1, since $k \geq 4$, Γ has at least 3 unfaithful types, say i, j, ℓ . Let $r \in I \setminus \{i, j, \ell\}$. By Lemma 5.4, $T_i^{X_r}, T_j^{X_r}$ and $T_\ell^{X_r}$ are all regular and $T_i^{X_r}, T_j^{X_r} \leq C_{\text{Sym}(X_r)}(T_\ell^{X_r})$. Lemma 2.1 implies that $T_i^{X_r} = C_{\text{Sym}(X_r)}(T_\ell^{X_r}) = T_j^{X_r}$. Moreover, by Lemma 5.4, $T_i^{X_r}$ and $T_j^{X_r}$ centralise each other, so $T_i^{X_r}$ is abelian. By Lemma 5.4(b), $T_i^{X_r} \cong T_i$, and the result follows. \square

Lemma 5.10. *Suppose that Γ and G satisfy Hypothesis 5.1, and assume also that $k = 3$, all types are G -unfaithful, and for some type s , T_s is non-abelian. Then the T_i are isomorphic nonabelian minimal normal subgroups of G , $\Gamma = \Gamma(3, m)$ with $m = |T_1|$, and $\langle T_1, T_2, T_3 \rangle = T_1 \times T_2 \times T_3$, as in line (ii) of Table 1.*

Proof. By Lemma 5.4 (a), $\langle T_i, T_j \rangle = T_i \times T_j$ for all i, j . Also, by Lemma 5.4(c) (and since T_s is nonabelian), $T_i \cong T_s$ is nonabelian for all i . By Lemma 5.4(b), for all $i \neq j$, T_i is regular and faithful on X_j . It follows that $\Gamma = \Gamma(3, m)$ with $m = |T_1|$. Let N be a minimal normal subgroup of G contained in T_i . Since Γ is G -normal-basic for $j \neq i$, N^{X_j} is transitive and as $T_i^{X_j}$ is faithful and regular on X_j it follows that $N = T_i$. Thus T_i is a minimal normal subgroup of G .

Let 1, 2, 3 be the three types and let $\hat{T}_3 = T_3 \cap (T_1 \times T_2)$. If $\hat{T}_3 \neq 1$ then as Γ is G -normal-basic, \hat{T}_3 is transitive on X_1 and X_2 and hence $|T_3| = |X_1|$ divides $|\hat{T}_3|$. Thus $T_3 = \hat{T}_3 \leq T_1 \times T_2$. Since T_1 and T_2 are direct products of nonabelian simple groups and are minimal normal subgroups of G , the only minimal normal subgroups of G contained in $T_1 \times T_2$ are T_1 and T_2 , a contradiction. Hence $\hat{T}_3 = 1$ and so $\langle T_1, T_2, T_3 \rangle = T_1 \times T_2 \times T_3$. \square

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. By Hypothesis 5.1, $|I_{\text{unf}}| \geq k - 1 \geq 2$. Assume first that $|I_{\text{unf}}| = 2$. Then since by assumption $k \geq 3$ and Γ has at most one faithful type, we have $k = 3$, and Lemma 5.6 shows that $|I_{\text{qp}}| = 1$, as in line (i) of Table 1.

Now suppose that $|I_{\text{unf}}| \geq 3$. Then Corollary 5.5 implies that $\Gamma = \Gamma(k, m)$ for some m . Assume first that $k = 3$, and let i be a type. If $T_i = G_{(X_i)}$ is non-abelian then Lemma 5.10 gives line (ii) of Table 1, and if T_i is abelian then Lemma 5.7 gives line (iii). Finally, assume that $k \geq 4$. Then Lemma 5.9 shows that the conditions of Lemma 5.7 hold, giving line (iii) again. \square

Proof of Theorem 1.2. By Proposition 4.3, either (i), (ii) or (iv) holds or Γ is complete multipartite and G is faithful on at most one of the X_i , that is, Hypothesis 5.1 holds. If $k = 2$ then $\Gamma = \Gamma(1, m) \oplus \Gamma(1, m')$ by Lemma 3.1, while if $k \geq 3$ then Proposition 5.3 yields the two remaining cases of the theorem. \square

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